Moore-Penrose inverse in Kreĭn spaces

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Abstract

We discuss the notion of Moore-Penrose inverse in Kreĭn spaces for both bounded and unbounded operators. Conditions for the existence of a Moore-Penrose inverse are given. We then investigate its relation with adjoint operators, and study the involutive Banach algebra $\mathfrak{B}(\mathcal{H})$. Finally applications to the Schur complement are given.

Keywords Moore-Penrose inverse, Kreĭn space, indefinite inner product, unbounded operators

Introduction

1 Kreĭn spaces, subspaces and operators

1.1 Definition

A Kreĭn space is an indefinite inner product space $(\mathcal{K}, [\cdot, \cdot])$ (*i.e.* the form $[\cdot, \cdot]$ is sesquilinear and hermitian) such that there exists an automorphism J of $\mathcal{K}$ which squares to the identity (called fundamental symmetry or signature operator), $\langle x, y\rangle \equiv [Jx, y]$ defines a positive definite inner product and $(\mathcal{K}, \langle\cdot, \cdot\rangle)$ is a Hilbert space. Equivalently, the indefinite inner product space $(\mathcal{K}, [\cdot, \cdot])$ is a Kreĭn space if there exist an admissible (with respect to the inner product) hilbertian topology on $\mathcal{K}$ that makes it an Hilbert space.

1.2 Positive, negative and neutral cone

The following subsets are defined in terms of the “square norm” induced by the indefinite inner product:

- $\mathbf{K}_+ \equiv \{x \in \mathcal{K} : [x, x] > 0\}$ is called the “positive cone”;
- $\mathbf{K}_- \equiv \{x \in \mathcal{K} : [x, x] < 0\}$ is called the “negative cone”;
- $\mathbf{K}_0 \equiv \{x \in \mathcal{K} : [x, x] = 0\}$ is called the “neutral cone”.

A subspace $\mathcal{L} \subset \mathcal{K}$ lying within $\mathbf{K}_0$ is called a ”neutral subspace”. Similarly, a subspace lying within $\mathbf{K}_+$ ($\mathbf{K}_-$) is called ”positive” (”negative”). A subspace in any of the above categories may be called ”semi-definite”, and any subspace that is not semi-definite is called ”indefinite”.

1.3 Kreĭn space decomposition and topology

Any decomposition of the indefinite inner product space $\mathcal{K}$ into a pair of subspaces $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ such that $\mathcal{K}_+ \subset \mathbf{K}_+ \cup \{0\}$ and $\mathcal{K}_- \subset \mathbf{K}_- \cup \{0\}$ is called a "fundamental decomposition" of $\mathcal{K}$. $\mathcal{K}_+$ equipped with the restriction of the bilinear form $[\cdot, \cdot]$ is then a Hilbert space, and $\mathcal{K}_-$ the antispace of Hilbert space $|\mathcal{K}_-|$. To this fundamental decomposition is associated a fundamental symmetry J such that the scalar product $\langle x, y\rangle \equiv [Jx, y]$ coincide with the scalar product of $|\mathrm{K}| = \mathcal{K}_+ \oplus |\mathcal{K}_-|$.
The positive definite inner product $\langle \cdot, \cdot\rangle$ depends on the chosen fundamental decomposition, which is, in general, not unique. But (see [5]) any two fundamental symmetries J and J' compatible with the same indefinite inner product on $\mathcal{K}$ result in Hilbert spaces $|\mathrm{K}|$ and $|\mathrm{K}'|$ whose decompositions $|\mathrm{K}|_\pm$ and $|\mathrm{K}'|_\pm$ have equal dimensions. Moreover they induce equivalent square norms hence a unique topology. This topology is admissible, and it is actually the Mackey topology defined by the bilinear pairing. All topological notions in a Kreĭn space, like continuity or closedness of sets are understood with respect to this Hilbert space topology.

1.4 Isotropic part and degenerate subspaces

Let $\mathcal{L}$ be a subspace of $\mathcal{K}$. The subspace $\mathcal{L}^{[\perp]} \equiv \{x \in \mathcal{K} : [x, y] = 0 \text{ for all } y \in \mathcal{L}\}$ is called the orthogonal companion of $\mathcal{L}$, and $\mathcal{L}^0 \equiv \mathcal{L} \cap \mathcal{L}^{[\perp]}$ is the isotropic part of $\mathcal{L}$. If $\mathcal{L}^0 = \{0\}$, $\mathcal{L}$ is called non-degenerate. It is called regular (or a Kreĭn subspace) if it is closed and a Kreĭn space with respect to the restriction of the indefinite inner product. This is equivalent to $\mathcal{L} \oplus \mathcal{L}^{[\perp]} = \mathcal{K}$ ([5]) and this relation is sometimes taken as a definition of regular subspaces. We will use the following lemma in the sequel (theorems 2.2 and 4.1 in [16] or theorem III.6.5. in [2]):

Lemma 1.1. *$\overline{\mathcal{L}}$ is non-degenerate* $\iff$ $\overline{\mathcal{L} + \mathcal{L}^{[\perp]}} = \mathcal{K}$.

In this case the sum is direct. We will also use the following identity: $\mathcal{L}^{[\perp]} = \overline{\mathcal{L}}^{[\perp]}$

1.5 Operators in Kreĭn spaces

If $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces, the space of densely defined operators from $\mathcal{H}$ into $\mathcal{K}$ will be denoted $\mathbf{OP}(\mathcal{H}, \mathcal{K})$. The subset of closed operators will be denoted by $\mathbf{C}(\mathcal{H}, \mathcal{K})$, the subset of bounded operators by $\mathbf{B}(\mathcal{H}, \mathcal{K})$. The subset of everywhere defined (not necessarily continuous) operators will be denoted by $\mathfrak{L}(\mathcal{H}, \mathcal{K})$ and the subset of everywhere defined continuous operators by $\mathfrak{B}(\mathcal{H}, \mathcal{K})$. Recall that bounded operators are defined with respect to the Hilbert norm induced by any fundamental decomposition.
We will note $D(A)$ for the domain of A, $N(A)$ for its null space of A and $R(A)$ for its range.

Let $A, B \in \mathbf{OP}(\mathcal{H}, \mathcal{K})$. A and B are adjoint (with respect to the indefinite inner product) if

$$\forall h \in D(A), \forall k \in D(B), \qquad [Ah, k]_{\mathcal{K}} = [h, Bk]_{\mathcal{H}}$$

If A is densely defined, then there exists a unique maximal adjoint denoted by A^*.

2 Moore-Penrose inverse in the Kreĭn space setting

2.1 Introduction

Recall that in the classical (matrix) setting the Moore-Penrose inverse $A^+ \in \mathbb{C}^{n \times m}$ of $A \in \mathbb{C}^{m \times n}$ is the unique linear operator which satisfies the following criteria:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^* = AA^+$
4. $(A^+A)^* = A^+A$

It always exists is unique, continuous and it is related to the minimal norm solution of a linear equation[18].
Moreover the Moore-Penrose inverse can be defined in the general setting of C^*-algebras, but then the existence is not guaranteed : $a \in A$ has a (unique) Moore-Penrose inverse if and only if it is (Von-Neumann) regular[7] ($a \in aAa$, a admits an inner inverse). Specializing to the operator algebra of a Hilbert space, this is equivalent to the closedness of $R(A)$ (operators verifying that property are sometimes called relatively regular).

If now $\mathcal{H}$ and $\mathcal{K}$ are Kreĭn spaces, then one can ask for the existence of a generalized inverse such that AA^+ and A^+A are self-adjoint (or symmetric in the unbounded case), but for the indefinite inner products. The question of the existence of a Moore-Penrose inverse in the setting, and its application to normal operators, appears for instance in[17].
Even when $\mathcal{H} = \mathcal{K}$, the indefiniteness of the inner product implies a major difference with the previous papers ([11], [7]): $\mathfrak{B}(\mathcal{H})$ fails to be a C^*-algebra, for the B^*-condition is not verified. Moreover, the Gelfand-Naimark property (valid for C^*-algebra) is not verified, which is the crucial property needed to prove that regularity implies Moore-Penrose invertibility.

Example 1. *Recall that the* B^*-*condition is*

$$\forall a \in A,\ ||a||^2 \le ||a^*a||$$

Let $\mathcal{H}$ *be the Euclidean space* $\mathbb{R}^2$ *with signature operator* $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. *Then it is straightforward to see that* $A^*A = 0$ *hence* $||A^*A|| = 0$, *but* $||A|| > 0$. *The* B^*-*condition is not satisfied.*
The B^*-*condition induces the Gelfand-Naimark property which itself is constitutive of symmetric* B^*-*algebras:*

$$\forall a \in A,\ e + a^*a \ \text{ is invertible.}$$

In the same Pontryagin space as before, let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. *Then* $B^* = -B$, $B^2 = I$ *and* $I + B^*B = 0$ *which proves that the Gelfand-Naimark property is not valid.*

Our main concern in the sequel will be to find necessary and sufficient conditions for a normalized generalized inverse B of an operator A to exist such that AB and BA are symmetric operators. We will first treat the case of densely defined (closed or not) operators, and then of everywhere defined (continuous or not) operators.

2.2 Generalized $*-$inverse for unbounded operators

Theorem 2.1. *Let* A *be an densely defined operator from* $\mathcal{H}$ *to* $\mathcal{K}$.
Then the two propositions are equivalent :

1. $\exists B \in \mathbf{OP}(\mathcal{K}, \mathcal{H})$ *such that :*
   (a) $R(A) \subset D(B);\ R(B) \subset D(A);$
   (b) $\forall h \in D(A),\ ABAh = Ah$
   (c) $\forall k \in D(B),\ BABk = Bk.$
   (d) AB *and* BA *are symmetric.*
2. $\overline{N(A)}$ *and* $\overline{R(A)}$ *are non-degenerate, and* $D(A) \subset N(A) \oplus N(A)^{[\perp]}$.

If B verifies (a), (b), (c), and (d), we call B a $*-$generalized inverse of A and following Labrousse[15] we note $A(* - inv)B$.

Proof. $\underline{\Rightarrow}$ i)Suppose $A(* - inv)B$ and let $h \in D(A)$. Then

$$h = (BA)h + (h - (BA)h)$$

with $(BA)h \in N(A)^{[\perp]}$ since BA is symmetric and $(h - (BA)h) \in N(A)$ since $ABA = A$ on $D(A)$. Then we have proved that

$$D(A) \subset N(A) + N(A)^{[\perp]}$$

and from the denseness of $D(A)$ we get

$$\overline{N(A) + N(A)^{[\perp]}} = \mathcal{H}$$

By lemma 1.1, the subspace $\overline{N(A)}$ is non-degenerate.

ii) Now let $k \in D(B)$. The existence of B allows the following decomposition

$$k = ABk + (k - ABk)$$

with $ABk \in R(A)$ and $(k - ABk) \in R(A)^{[\perp]}$ since AB is symmetric. It follows that $\overline{R(A) + R(A)^{[\perp]}} = \mathcal{K}$ since $D(B)$ is dense and by lemma 1.1, $\overline{R(A)}$ is non-degenerate.

$\Leftarrow$ Suppose $\overline{N(A)}$ and $\overline{R(A)}$ are non-degenerate. Then $\mathcal{H} = \overline{N(A) \oplus N(A)^{[\perp]}}$ and $\mathcal{K} = \overline{R(A) \oplus R(A)^{[\perp]}}$ by lemma 1.1. Since $D(A) \subset N(A) \oplus N(A)^{[\perp]}$

$$A_{|\left(N(A)^{[\perp]} \bigcap D(A)\right)} : \left(N(A)^{[\perp]} \cap D(A)\right) \to R(A)$$

is bijective. The following operator

$$\begin{array}{rcc} B : D(B) = R(A) \oplus R(A)^{[\perp]} & \longrightarrow & \mathcal{H} \\ k + k' & \longmapsto & (A_{|N(A)^{[\perp]} \bigcap D(A)})^{-1}(k) \end{array}$$

verifies the desired properties. □

The operator B of the proof seems to play a special role among the $*-$generalized inverses of A. The following lemmas show that its domain is maximal among the $*-$generalized inverses.

Lemma 2.2. *Suppose $A(* - inv)B$. Then :*

1. *$B(* - inv)A$, A is a $*-$generalized inverse of B.*
2. *$N(A) \subset R(B)^{\perp}$ and $N(B) \subset R(A)^{\perp}$.*
3. *AB is a symmetric projection from $D(B) = R(A) \oplus N(B)$ on $R(A)$, with kernel $N(B)$.*
4. *BA is a symmetric projection from $D(A) = R(B) \oplus N(A)$ on $R(B)$, with kernel $N(A)$.*

These results follow directly from the properties of a $*-$generalized inverse and lemma 1.3[15].
The existence of a unique maximal $*-$generalized inverse is now given by the following theorem:

Theorem 2.3. *Let $A \in \mathbf{OP}(\mathcal{K}, \mathcal{H})$ such that $\overline{N(A)}$ and $\overline{R(A)}$ are non-degenerate and $D(A) \subset N(A) \oplus N(A)^{[\perp]}$. Define*

$$\begin{array}{rcl} A^+ : D(A^+) = R(A) \oplus R(A)^{[\perp]} & \longrightarrow & \mathcal{H} \\ k + k' & \longmapsto & (A_{|N(A)^{[\perp]} \bigcap D(A)})^{-1}(k) \end{array}$$

Then $A(* - inv)B \Rightarrow B$ is a restriction of A^+.

Proof. Let A^+ and B be as defined in the theorem. Then $\forall h, h' \in D(A)$:

$$\begin{aligned} \left[h, A^+Ah'\right]_{\mathcal{H}} &= \left[A^+Ah, h'\right]_{\mathcal{H}} = \left[A^+ABAh, h'\right]_{\mathcal{H}} = \left[BAh, A^+Ah'\right]_{\mathcal{H}} \\ &= \left[h, BAA^+Ah'\right]_{\mathcal{H}} = [h, BAh']_{\mathcal{H}} \end{aligned}$$

and since $D(A)$ is dense in $\mathcal{H}$

$$\forall h \in D(A),\ A^+Ah = BAh$$

From the previous results, $D(B) = N(B) \oplus R(A) \subset R(A)^{[\perp]} \oplus R(A) = D(A^+)$. it follows that $\forall k, k' \in D(B)$:

$$\begin{aligned} \left[k, AA^+k'\right]_{\mathcal{H}} &= \left[AA^+k, k'\right]_{\mathcal{K}} = \left[ABAA^+k, k'\right]_{\mathcal{K}} = \left[AA^+k, ABk'\right]_{\mathcal{K}} \\ &= \left[k, AA^+ABk'\right]_{\mathcal{K}} = [k, ABk']_{\mathcal{K}} \end{aligned}$$

and since $D(B)$ is dense in $\mathcal{K}$ we get

$$\forall k \in D(B),\ AA^+k = ABk$$

Finally

$$\forall k \in D(B),\ A^+k = A^+AA^+k = BAA^+k = BABk = Bk$$

and A^+ is an extension of B. □

We call this maximal operator A^+ the maximal inverse of A, or Moore-Penrose inverse of A.

Lemma 2.4. *Suppose A^+ exists.*

1. $D(A^+) = R(A) \oplus R(A)^{[\perp]}$, $N(A^+) = R(A)^{[\perp]}$, $R(A^+) = N(A)^{[\perp]} \cap D(A)$
2. *$(A^+)^+$ exists.*
3. *If A is everywhere defined, then $N(A)$ is regular.*
4. *A^+ is everywhere defined $\iff$ $R(A)$ is regular.*
5. *$(A^+)^+ = A \Rightarrow N(A)$ is closed.*
6. *$N(A)$ regular $\Rightarrow (A^+)^+ = A$.*

Proof. Suppose A^+ exists.

1. Follows from the definition of A^+.

2. Let us prove that A^+ satisfies the conditions of theorem 2.1. First $N(A^+) = R(A)^{[\perp]}$ and $N(A^+)$ is non-degenerate. Second $D(A^+) = R(A) \oplus R(A)^{[\perp]} \subset N(A^+)^{[\perp]} \oplus N(A^+)$ from the same equality. Finally,
$$D(A) = N(A) + N(A)^{[\perp]} \cap D(A) \subset R(A^+)^{[\perp]} + R(A^+)$$
and it follows from the denseness of $D(A)$ and lemma 1.1 that $\overline{R(A^+)}$ is non-degenerate.

3. by theorem 2.1 the existence of a $*-$ generalized inverse implies $D(A) \subset N(A) \oplus N(A)^{[\perp]}$ and the result follows.

4. Follows from the equality $D(A^+) = R(A) \oplus R(A)^{[\perp]}$.

5. Since A is a $*-$generalized inverse of A^+, A is a restriction of $(A^+)^+$. By definition of $(A^+)^+$, $D((A^+)^+) = R(A^+) \oplus (R(A^+))^{[\perp]}$ and by lemma 2.2 $D(A) = R(A^+) \oplus N(A)$, $N(A) \subset R(A^+)^{[\perp]}$. $(A^+)^+ = A \Rightarrow D(A) = D((A^+)^+) \Rightarrow N(A) = (R(A^+))^{[\perp]}$ and $N(A)$ is closed.

6. Suppose $N(A)$ regular. Then any $h \in R(A^+)^{[\perp]}$ admits a decomposition $h = h_1 + h_2$ with $h_1 \in N(A)$ and $h_2 \in N(A)^{[\perp]}$. Then $h_2 = h - h_1 \in R(A^+)^{[\perp]} \cap N(A)^{[\perp]} = \left(R(A^+) \oplus N(A)\right)^{[\perp]} = \{0\}$ since A is densely defined. Finally $R(A^+)^{[\perp]} = N(A)$ and $(A^+)^+ = A$.

□

Suppose now that A is closed. Its seems then natural to restrict our attention to closed $*-$generalized inverses. Next theorem gives a sufficient condition for A^+ to be closed:

Theorem 2.5. *Let $A \in \mathbf{C}(\mathcal{H}, \mathcal{K})$ admits a Moore-Penrose inverse.*
If $\overline{R(A)}$ is regular, then A^+ is closed.

Proof. First, since A is closed $N(A)$ is closed, and since $\overline{R(A)}$ is regular it is non-degenerate. Hence by the previous theorems
$$\begin{array}{rcl} A^+ : D(A^+) = R(A) \oplus R(A)^{[\perp]} & \longrightarrow & \mathcal{H} \\ k + k' & \longmapsto & (A_{|N(A)^{[\perp]} \bigcap D(A)})^{-1}(k) \end{array}$$
is well defined.
Let $k_n \in D(A^+) = R(A) \oplus R(A)^{[\perp]}$ converge towards $k \in \mathcal{K}$, such that $A^+ k_n \to h \in \mathcal{H}$. Then $h \in N(A)^{[\perp]}$ since $h \in \overline{R(A^+)}$ and $R(A^+) \subset N(A)^{[\perp]}$.
Denote by P the orthogonal projection on $\overline{R(A)}$ and let $k_n = Au_n + v_n$ be the decomposition of $k_n \in R(A) \oplus R(A)^{[\perp]}$, $Au_n = Pk_n$.Then $A^+k_n = A^+Au_n = u_n \to h$ and $AA^+k_n = Au_n = Pk_n \to Pk$ since $k_n \to k$ by hypothesis and P

is continuous. The closedness of A then implies: $h \in D(A), Ah = Pk$. Finally $Pk \in R(A)$, $k = Pk + (I-P)k \in D(A^+)$ and

$$A^+k = A^+(Pk + (I-P)k) = A^+Pk = A^+Ah = h$$

since $h \in N(A)^{[\perp]}$. Finally A^+ is closed. □

Theorem 2.6. *Let $A, B \in \mathbf{C}(\mathcal{H}, \mathcal{K})$ such that $A(*-inv)B$. Suppose moreover that $\overline{R(A)}$ is regular. Then:*

1. *B is bounded $\Rightarrow R(A)$ is closed.*
2. *B is bounded $\Rightarrow B$ is everywhere defined and $B = A^+$.*
3. *$R(A)$ is closed $\Rightarrow A^+$ is everywhere defined and bounded, $A^+ \in \mathfrak{B}(\mathcal{K}, \mathcal{H})$.*

Proof.

1. Suppose B is bounded. Since B is closed, $D(B) = R(A) \oplus N(B)$ is closed and by proposition 2.11 in [14], $R(A)$ is closed.
2. Moreover $D(B) = \mathcal{K}$ since it is closed and dense. The equality $B = A^+$ follows.
3. If now $R(A)$ is closed, then $D(A^+) = R(A) \oplus R(A)^{[\perp]} = \mathcal{K}$ is closed and by the closed graph theorem, A^+ is bounded.

□

2.3 Generalized $*-$inverse and regular operators

It follows from the previous theorems that the regularity of $N(A)$, as well as $\overline{R(A)}$ seems to play a great role, particularly for closed (or everywhere defined) operators. Actually, if one starts from A with $N(A)$ regular, then $N(A^+)$ will be regular if and only if $\overline{R(A)}$ is regular (note that these subspaces are equal by definition of A^+).

Definition 2.7. *We say that an operator $A \in \mathbf{OP}(\mathcal{H}, \mathcal{K})$ is weakly regular if $N(A)$ and $\overline{R(A)}$ are regular. We note the set of weakly regular operators* $\mathbf{WR}(\mathcal{H}, \mathcal{K})$.

Theorem 2.8. *Let $A \in \mathbf{OP}(\mathcal{H}, \mathcal{K})$.*

1. *If $A \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$, then it is $*-$invertible, and $A^+ \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$.*
2. *If $B \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$ is a $*-$inverse of A, Then $B = A^+$.*

This first sentence read: “Any weakly regular operator admits a weakly regular $*-$generalized inverse” whereas the second one read “if there exist a weakly regular $*-$generalized inverse, it is unique.” Remark that by lemma 2.4 (or directly by unicity), for weakly regular operators $A = (A^+)^+$.

Proof. 1. First, $A \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$ verifies the condition of theorem 2.1 and A^+ exists.

Second, $N(A^+) = \overline{R(A)}$ and $N(A^+)$ is regular.

Third, $R(A^+) = N(A)^{[\perp]} \cap D(A)$. Let $x \in N(A)^{[\perp]}$, $\forall y \in R(A^+)$, $[x, y] = 0$. Then $\forall z = z_1 + z_2 \in D(A) \subset N(A) \oplus N(A)^{[\perp]}$, $z_2 = z - z_1 \in D(A)$ and $[x, z] = [x, z_1 + z_2] = [x, z_2] = 0$ by hypothesis. it follows that$R(A^+)$ is dense in the Kreĭn subspace $N(A)^{[\perp]}$, and $\overline{R(A^+)} = N(A)^{[\perp]}$.

2. Let now $B \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$ be an other $*-$generalized inverse. Then B is a restriction of A^+ with domain $D(B) = N(B) \oplus R(A)$. Since $D(A^+) = R(A)^{[\perp]} \oplus R(A)$ we have to prove that $N(B) = R(A)^{[\perp]}$ (note that we already know that $N(B) \subset R(A)^{[\perp]}$). Since $B \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$, $N(B)$ is regular and $N(B) \oplus N(B)^{[\perp]} = \mathcal{K}$. Let $k = k_1 + k_2 \in R(A)^{[\perp]} \subset N(B) \oplus N(B)^{[\perp]}$. Then $k_2 = k - k_1 \in R(A)^{[\perp]} \cap N(B)^{[\perp]}$. But $R(A)^{[\perp]} \cap N(B)^{[\perp]} = (R(A) + N(B))^{[\perp]}$ ([5]) and $h_2 = 0$ since $D(B) = N(B) \oplus R(A)$ is dense in $\mathcal{K}$. Finally $R(A)^{[\perp]} \subset N(B)$ and $B = A^+$. □

Corollary 2.9. *Let A be closed and weakly regular. Then it admits a unique closed and weakly regular $*-$inverse, its Moore-Penrose inverse A^+.*

Remark 2.10. *Weakly regular operators may be connected with the notion of strict generalized inverse of Labrousse[15]. Actually, one can prove that an operator A admits a strict generalized $*-$inverse if and only if it is weakly regular, and in this case, the inverse is weakly regular, hence unique and precisely A^+.*

Finally, we specialize to the case of everywhere defined operators.

Definition 2.11. *We say that an operator $A \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$ is regular if $N(A)$ and $R(A)$ are regular. We note the set of regular operators $\mathbf{R}(\mathcal{H}, \mathcal{K})$.*

Theorem 2.12. *Let $A \in \mathfrak{L}(\mathcal{H}, \mathcal{K})$.*

$$\exists B \in \mathfrak{L}(\mathcal{H}, \mathcal{K}), A(* - inv)B \iff A \text{ is regular.}$$

If it exists, A^+ is continuous $\iff$ A is continuous.

Proof. $\Rightarrow$ Let B is a $*-$generalized inverse everywhere defined. Then $B = A^+$ since its domain is maximal. It first follows that such an inverse is unique. Second by lemma 2.4, $N(A)$ and $R(A)$ are regular.

$\Rightarrow$ Suppose now that A is regular. Then by theorem 2.8 it is invertible and $A^+ \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$. But $D(A^+) = R(A)^{[\perp]} \oplus R(A) = \mathcal{K}$ since $R(A)$ is regular, and A^+ is everywhere defined.

If A is continuous, it is closed and by theorem 2.6, A^+ is continuous. Conversely, if A^+ is continuous, then $A = (A^+)^+$ (lemma 2.4) is continuous by the previous argument. □

Corollary 2.13. *Let A be continuous and regular. Then*

1. *$AA^+ : \mathcal{K} \to \mathcal{K}$ is a self-adjoint projection on $R(A)$ with kernel $N(A^+) = R(A)^{[\perp]}$.*

2. *$A^+A : \mathcal{H} \to \mathcal{H}$ is a self-adjoint projection on $R(A^+) = N(A)^{[\perp]}$, with kernel $N(A)$.*

Remark 2.14. *The conditions for A to admit Moore-Penrose inverse involve the existence of a certain decomposition of the space int a direct sum of subspaces. Such type of decomposition arise also in the setting of $C*-$algebras when studying commuting generalized inverses. If A admits a commuting generalized inverse, then $\mathcal{H} = R(A) \oplus N(A)$ ([7] Theorem 9).*

2.4 Adjoint of regular operators

We focus now on closed densely defined operators (or on continuous operators). Then it is a classical result that A^* is also a closed densely defined operator and that $(A^*)^* = A$. The relations between regularity of A and A^* are described in the following theorem:

Theorem 2.15. *Let $A \in \mathbf{C}(\mathcal{H}, \mathcal{K})$. Then $A \in \mathbf{WR}(\mathcal{H}, \mathcal{K}) \iff A^* \in \mathbf{WR}(\mathcal{H}, \mathcal{K})$*
and in this case $(A^*)^+ = (A^+)^*$.
If moreover $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$, then $A \in \mathbf{R}(\mathcal{H}, \mathcal{K}) \iff A^* \in \mathbf{R}(\mathcal{H}, \mathcal{K})$.

Proof. The equivalences are a direct consequence of the following equality:

$$N(A^*) = R(A)^{[\perp]}$$

and the equivalence $R(A^*)$ *closed* $\iff$ $R(A)$ *closed.*
For the equality $(A^*)^+ = (A^+)^*$, note that $(A^+)^*$ is a $*-$generalized inverse of A^*. We conclude by unicity of the inverse in the set of (weakly) regular operators (theorem 2.8). □

Then theorem 2.12 can be rewritten in the following form (where the implication and the decompositions were known to be true in finite-dimensional Pontryagin spaces[10]):

Theorem 2.16. *Let $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$. The following statements are equivalent:*

1. *A admits a (unique) Moore-Penrose inverse.*

2. *$N(A) \oplus R(A^*) = \mathcal{H}$ and $R(A) \oplus N(A^*) = \mathcal{K}$.*

In this case there are equalities:

$$R(A^+) = R(A^*) \text{ and } N(A^+) = N(A^*)$$

and the following decompositions hold:

$$N(A) \oplus R(A^+) = \mathcal{H} \text{ and } R(A) \oplus N(A^+) = \mathcal{K}$$

Proof. ⇒ Suppose A admits a Moore-Penrose inverse. Then (theorem 2.12) A is regular and by theorem 2.15 A^* is regular and finally $N(A)$, $R(A)$, $N(A^*)$ and $R(A^*)$ are regular. But $N(A)^{[\perp]} = \overline{R(A^*)} = R(A^*)$ since $R(A^*)$ is regular hence closed. But also $N(A^*) = R(A)^{[\perp]}$ and the decomposition follows.

⇐ Suppose now $N(A) \oplus R(A^*) = \mathcal{H}$ and $R(A) \oplus N(A^*) = \mathcal{K}$. Since $N(A^*) = R(A)^{[\perp]}$, $R(A)$ is regular. For $N(A)$, remark that $R(A^*) \subset N(A)^{[\perp]}$ implies $N(A) + N(A)^{[\perp]} = \mathcal{H}$. But by lemma 1.1 this also implies that $N(A)$ is non-degenerate, and finally A is regular.

The other equalities follow from the definition of A^+. □

Theorem 2.12 gives an elegant proof a the following theorem due to McEnnis (Theorem 5.3 in[16]) about the continuity of isometries in Kreĭn spaces:

Theorem 2.17. *Suppose that $V : \mathcal{H} \to \mathcal{K}$ satisfies $[Vh, Vl] = [h, l]$ $\forall h, l \in \mathcal{H}^2$. Then V is continuous if and only if $R(V)$ is regular.*

Proof. Suppose V is continuous. Then its adjoint V^* exists and satisfies the Moore-Penrose inverse properties: $V^+ = V^*$. By theorem 2.12, $R(V)$ is regular. Suppose now $R(V)$ is regular. By polarity V is injective and $N(V) = \{0\}$ is regular. It follows that V^+ exists and $\forall h \in \mathcal{H}, k \in \mathcal{K}$

$$[Vh, k] = \left[VV^+Vh, k\right] = \left[Vh, VV^+k\right] = \left[h, V^+k\right]$$

and $V^* = V^+$. It follows that V is weakly continuous hence continuous. □

Many theorems on generalized inverse rely on the products AA^* and A^*A. They will be shown to be true under the additional hypothesis of $*-$cancelation.

Definition 2.18. $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ *is $*-$cancelable if*

$$A^*Ah = 0 \Rightarrow Ah = 0$$

(or equivalently $N(A^*A) = N(A)$) and

$$AA^*k = 0 \Rightarrow A^*k = 0$$

(or equivalently $N(AA^*) = N(A^*)$).

We start with a simple lemma:

Lemma 2.19. *Let $A \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ be regular. Then it is $*-$cancelable.*

Proof. Since A is regular, A^+ exists and by theorem 2.15 $(A^*)^+ = (A^+)^*$. Suppose $A^*Ah = 0$. Then by corollary 2.13, AA^+ is self-adjoint and finally

$$Ah = AA^+Ah = (AA^+)^*Ah = (A^+)^*A^*Ah = 0$$

But by theorem 2.15 A^* is also regular and by the same arguments, $N(AA^*) = N(A^*)$. □

The main theorem of this section follows:

Theorem 2.20. *Let* $A \in \mathfrak{B}(\mathcal{H},\mathcal{K})$.
A is regular $\iff$ A^*A *is regular and* A *is* $*-$*cancelable* $\iff$ AA^* *is regular and* A^* *is* $*-$*cancelable.*
In this case

1. $(A^*A)^+ = A^+(A^+)^*$
2. $(AA^*)^+ = (A^+)^*A^+$
3. $A^+ = (A^*A)^+A^* = A^*(AA^*)^+$

Proof. We will only prove the first equivalence, for the other one, take the adjoint and use the equality $(A^*)^+ = (A^+)^*$.
For the implication, $B = A^+(A^+)^*$ verifies $A^*A(*-inv)B$ and by theorem 2.12 A^*A is regular. A is $*-$cancelable by lemma 2.19.
Suppose now A^*A is regular and A is $*-$cancelable. Then $B = (A^*A)^+A^*$ exists. We have

$$\begin{aligned} (AB)^* &= B^*A^* = A\left((A^*A)^+\right)^* A^* = A\left((A^*A)^*\right)^+ A^* = AB \\ (BA)^* &= A^*B^* = A^*A\left(A^*A)^+\right)^* = \left((A^*A)^+A^*A\right)^* = (A^*A)^+A^*A = BA \\ BAB &= (A^*A)^+A^*A(A^*A)^+A^* = (A^*A)^+A^* = B \\ A^*ABA &= A(A^*A)^+A^*A = A^*A \end{aligned}$$

and by $*-$cancelation, $ABA = A$.
Finally A is Moore-Penrose invertible hence regular and $A^+ = (A^*A)^+A^*$ by unicity of the Moore-Penrose inverse (theorem 2.8). ☐

The $*-$cancelation hypothesis is necessary. Let once again $\mathcal{H}$ be the Euclidean space $\mathbb{R}^2$ with signature operator $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Then when have seen that $A^*A = 0$ and A^*A is regular, but A is not. Remark that A is not $*-$cancelable. Once again, this is very different from the situation on C^*-algebras where $*-$cancelation is always verified and A^*A regular implies A regular (theorem 7 in [7]).
In case of finite dimensional Kreĭn spaces or if A has a finite dimensional range, this reduces to a simple equality of ranks:

Corollary 2.21. *Let* $A \in \mathfrak{L}(\mathcal{H},\mathcal{K})$ *and suppose* $R(A)$ *is finite dimensional. Then*

$$A \text{ is regular } \iff rank(A) = rank(A^*A) = rank(AA^*)$$

Proof. The implication follows from theorem 2.20. For the converse implication, $rank(A) = rank(A^*A) \Rightarrow R(A^*) = R(A^*A)$ and by polarity, $N(A) = N(A^*A)$. Symmetrically, $N(A^*) = N(AA^*)$, and finally A is $*-$cancelable. Remains to prove that $N(A^*A) \oplus N(A^*A)^{[\perp]} = \mathcal{H}$ and $R(A^*A) \oplus R(A^*A)^{[\perp]} = \mathcal{H}$. Once again by polarity, and since $R(A^*A)$ is closed these two equalities involve exactly the same subspaces and we have only to prove that $N(A^*A) \oplus R(A^*A) = \mathcal{H}$. Since $R(A^*A)$ is finite dimensional, this reduces to prove that

$$N(A^*A) \cap R(A^*A) = \{0\}$$

Let $x = A^*Ay \in N(A^*A) \cap R(A^*A)$. Then

$$0 = A^*Ax = A^*AA^*Ay = AA^*Ay = A^*Ay = x$$

by $*-$cancelation. It follows that A^*A is regular and A is $*-$cancelable and by theorem 2.20, A is regular. □

This equivalence appears for the first time in a work of Kalman[9] who studies weighted inverses of matrices.

3 Moore-Penrose inverse in the $^*-$Banach algebra $\mathfrak{B}(\mathcal{H})$

When $\mathcal{H} = \mathcal{K}$, the algebra of continuous linear operators $\mathfrak{B}(\mathcal{H})$ is a $^*-$Banach algebra (hence a ring with involution). As noted before, the fact $\mathfrak{B}(\mathcal{H})$ is not symmetric implies major differences with classical works[13]. However, all properties of the Moore-Penrose inverse that do not rely on the Gelfand-Naimark property, but only on algebraic calculations remain.
For instance, the application to commuting operators (and in particular to normal operators) follows directly from the results of [7], [8] and [12], that do not rely on the Gelfand-Naimark property.

Proposition 3.1 ([7], theorem 5)**.** *If A is regular, then $A^+ \in comm^2(A, A^*)$ (where $comm^2(\mathcal{A})$ denote the double commutant of $\mathcal{A}$).*

Proposition 3.2 ([7], theorem 10)**.** *If A is regular and normal, then A^+ is also normal and $A^+ \in comm(A, A^*)$ (where $comm(\mathcal{A})$ denote the commutant of $\mathcal{A}$).*

Proposition 3.3 ([12], proposition 2.13)**.** *If A, B are regular and $B \in comm(A, A^*)$, then AB is regular and*

$$A^+B = BA^+, = AB^+ = B^+A, \ (AB)^+ = A^+B^+ = B^+A^+$$

Finally, it is well known that there exists at most one commuting generalized inverse ([7] Theorem 9). We may wonder when this inverse is precisely A^+, *i.e.* when $AA^+ = A^+A$. This question has already been studied in the context of matrix[19], of bounded operators on a Hilbert space[3] or more generally on C^*-algebras[8]. Their conditions are actually the same in our $^*-$Banach algebra $\mathfrak{B}(\mathcal{H})$:

Proposition 3.4. *Let $A \in \mathfrak{B}(\mathcal{H})$ be regular. The following propositions are equivalent:*

1. $A^+A = AA^+$
2. $N(A)^{[\perp]} = R(A)$
3. $N(A) = N(A^*)$
4. $R(A) = R(A^*)$
5. $\exists T \in \mathfrak{B}(\mathcal{H})$ *invertible,* $A = TA^*$
6. $\exists T \in \mathfrak{B}(\mathcal{H})$ *invertible,* $A = A^*T$

Proof. From lemmas 2.2 and 2.4, A^+A is the orthogonal projection on $R(A^+) = N(A)^{[\perp]}$ and AA^+ is the orthogonal projection on $R(A)$. hence the equivalence between 1) and 2) is proved. Statements 3) and 4) follow from the closedness of all the subspaces involved and the equality

$$N(A^*) = R(A)^{[\perp]}$$

For the last two statements, we refer to the calculations of [8] Theorem 10. □

4 Moore-Penrose inverse and Schur Complement

Of particular interest in the theory of Kreĭn spaces are linear systems[1], together with transfer functions and Schur theory. It is then of interest to study the Moore-Penrose invertibility of block operators. This section relies mainly on the theorems and calculations in [6]. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, and let $M \in \mathfrak{B}(\mathcal{H}, \mathcal{K})$ have the following matrix representation with respect to these decompositions:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

If A is regular, then A^+ exists and we can define the (generalized) Schur complement of A in M[4]

$$(M/A) = D - CA^+B$$

and if D is regular, then D^+ exists and we can define the Schur complement of D in M

$$(M/D) = A - BD^+C$$

Of course (M/A) (resp. (M/D)) need not be regular. Suppose it is. Then theorem 1 and corollary 1 and 2 in [6] are valid, for they rely only on basic matrix calculations:

Theorem 4.1. *Suppose A and M/A are regular and the following conditions (N) hold:*

1. $N(A) \subset N(C)$*;*
2. $N(A^*) \subset N(B^*)$*;*

3. $N(M/A) \subset N(B)$;
4. $N((M/A)^*) \subset N(C^*)$;

Then M is regular and

$$M^+ = \begin{pmatrix} A^+ + A^+B(M/A)^+CA^+ & -A^+B(M/A)^+ \\ -(M/A)CA^+ & (M/A)^+ \end{pmatrix}$$

In this case the Schur complement of $(M/A)^+$ in M^+ is regular with Moore-Penrose inverse A.

Of course the same theorem holds considering the Schur complement of D in M.

Theorem 4.2. *Suppose D and M/D are regular and the following conditions (N') hold:*

1. $N(D) \subset N(B)$;
2. $N(D^*) \subset N(C^*)$;
3. $N(M/D) \subset N(C)$;
4. $N((M/D)^*) \subset N(B^*)$;

Then M is regular and

$$M^+ = \begin{pmatrix} (M/D)^+ & -(M/D)BD^+ \\ -D^+C(M/D)^+ & D^+ + D^+C(M/D)^+BD^+ \end{pmatrix}$$

In this case the Schur complement of $(M/D)^+$ in M^+ is regular with Moore-Penrose inverse D.

Corollary 4.3. *Suppose A, M/A D M/D regular and conditions (N), (N') hold. Then M is regular and*

$$M^+ = \begin{pmatrix} (M/D)^+ & -A^+B(M/A)^+ \\ -D^+C(M/D)^+ & (M/A)^+ \end{pmatrix}$$

Remark that by unicity of the Moore-Penrose inverse, under the hypothesis of the previous corollary we get the equalities:

$$(M/A)^+ = D^+ + D^+C(M/D)^+BD^+$$

$$(M/D)^+ = A^+ + A^+B(M/A)^+CA^+$$

Suppose now that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ are Hilbert space, and the Kreĭn space structure is given by the block signature operator

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the operators A, M/A D M/D are regular if and only if their range is closed. Conditions (N), (N') are then sufficient for M to be regular as an operator between Kreĭn spaces.

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